Definition of Stress

• Stress, $\sigma$, is defined as the intensity of force at a point:
  \[ \sigma = \frac{dF}{dA} \text{ as } dA \text{ approaches 0} \]

• If the state of stress is the same everywhere in a body,
  \[ \sigma = \frac{F}{A} \]

• Stress can be classified as “normal” stresses and “shear” stresses
Units

• The International System of Units (SI units) is usually adopted.

• The SI units of force is the newton (N). Stresses and pressures in SI units are in newtons per square meter, N/m², which is given the special name of pascal (Pa). Millions of pascal (megapascals, MPa) are generally appropriate for our use, i.e.,

\[
1 \text{ MPa} = 1 \text{ MN/m}^2 = 1 \text{ N/mm}^2
\]

Sign Convention

• Tensile “normal” stress as positive sign, while compressive “normal” stress as negative sign. For shear stress, see the plot on LHS.

Positive shear
Energy Potential b/t a Cation-Anion Pair (Example)

- Strong primary chemical bonding between atoms/ions are resistant to stretching and so result in a high strength of materials.
Strengths of Irons and Steels

• The ultimate tensile strength ($\sigma_u$) for irons and steels in various forms.

Effect of Alloying on Strength of Steels

• The yield strength ($\sigma_y$ or $\sigma_p$) of steels is often affected by the alloying additions, alloying concentrations, and processing variables as well.
Effect of Microstructure on Strength of Coppers

- Nanocrystalline Cu exhibits a yield strength exceeding 400 MPa, 6 times higher than that of microcrystalline Cu. But, nano-Cu is often brittle.
- Y. Wang et al. have found a way to overcome the above problem by tailoring the microstructure with a bimodal (nano plus micro) grain-size distribution (Y. Wang et al., Nature, 419 (2002) 912)).

Strength of Selected Metals

- Here is a list of properties of selected engineering metals and their alloys. Note that strength, in particular, may vary as the alloying (purity) level, processing variables change.
Tensor Notation of Stress

- Nine components of stresses are needed to describe a state of stress at a point fully. As shown in the figure, \( \sigma_{zy} \) (or \( \tau_{zy} \)) is the stress caused by a shear force in the y direction acting on a plane normal to z.

- In tensor notation, the state of stress at a point is expressed as

\[
\sigma_{ij} = \begin{bmatrix}
\sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\
\sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\
\sigma_{zx} & \sigma_{zy} & \sigma_{zz}
\end{bmatrix},
\]

where \( i \) and \( j \) are iterated over x, y, and z. Note that stress tensor is symmetrical, \( \Rightarrow \) there are 6 independent variables instead of 9.

Shear Stress

- Most of the engineering materials are particularly vulnerable to shear stresses. In fact, the materials listed in the previous viewgraph are most likely failed by shear.

- Definition of shear stress (\( \tau \)) is same as that of the normal stress (\( \sigma \)), but \( \tau \) is applied on surface in parallel direction

\[
\tau = \frac{F}{A}
\]

- Shear stress always comes with a pair (because of force/moment balance in statics); in addition, the shear stresses with a reversed subscripts are always equal.

\[
\tau_{xy} = \tau_{yx} \quad \text{or} \quad \tau_{ij} = \tau_{ji}
\]
**Example**

**EXAMPLE 2.12**

Two timbers, of cross-sectional dimension $b \times h$, are to be glued together using a tongue-and-groove joint as shown in Fig. 2.36, and we wish to estimate the depth $d$ of the glue joint so as to make the joint approximately as strong as the timber itself.

The axial load $P$ on the timber acts to shear the glue joint, and the shear stress in the joint is just the load divided by the total glue area:

$$\tau = \frac{P}{2bd}$$

If the bond fails when $\tau$ reaches a maximum value $\sigma_r$, the load at failure will be $P_r = (2bd)\sigma_r$. The load needed to fracture the timber in tension is $P_r = bh\sigma_l$, where $\sigma_l$ is the ultimate tensile strength of the timber. Hence, if the glue joint and the timber are to be equally strong, we have

$$(2bd)\sigma_r = bh\sigma_l \rightarrow d = \frac{h\sigma_l}{2\sigma_r}$$

**Figure 2.36** Tongue-and-groove adhesive joint.

---

**From Simple to Complex State of Stress**

- In practice, components of any structures are subjected to applied loadings that may include tension, compression, torsion, bending, etc. As a result, complex states of normal and shear stress occur that vary in magnitude and direction with location in the component.

- Fortunately, a methodology has been developed to find out the directions at which the state of stresses is the most severe. The stresses involved are called *principal stresses*, and the particular directions in which they act are the *principal axes*.
From Simple to Complex State of Stress

- **Plane stress** is one of a simpler case in complex state of stress. It may be simply defined as: **there is a plane (or direction) without stress**. This occurs at any free (unloaded) surface, and surface locations often have the most severe stresses, as in *thin films* loaded in any form of stress.

- For an x-y-z coordinate system, a small element of material (see the previous viewgraph) is subjected to six components of stress. For a state of plane stress
  \[ \sigma_z = \tau_{zx} = \tau_{zy} = 0 \]

Example

For a thin-walled sphere, circumferential stresses involved in:

\[ P \cdot \hat{r} = \sigma_r \cdot 2\pi r \]

- \( r \): sphere radius
- \( b \): wall thickness

Remark:

- \( \sigma_r \) at the surface can be neglected if \( r \gg b \), i.e., plane stress condition.
From Simple to Complex State of Stress

- (continue on plane stress) Equilibrium of forces on the element requires that the moments must sum to zero about both the x- and y-axes; therefore, $\tau_{zx}$ and $\tau_{zy}$ acting on the other two planes must also be zero.
- Hence, the components remaining are $\sigma_x$, $\sigma_y$, and $\tau_{xy}$. Shown below is one example when all stresses are positive.

Note: z-axis is perpendicular to the plane.

Rotation of Coordinate Axes (2D)

- The same state of plane stress may be described on any other coordinate system, such as $x'$-$y'$ with an angle of rotation $\theta$, and the values of the stress components change accordingly to $\sigma'_{x}$, $\sigma'_{y}$, and $\tau_{xy}'$ in the new coordinate system.
Rotation of Coordinate Axes (2D)

Now we derive a general strain state including \( \sigma_x', \sigma_y', \text{ and } \gamma_{xy}' \).

\[
\begin{align*}
\mathbf{\sigma}' &= \mathbf{R} \mathbf{\sigma} \mathbf{R}^T \\
\mathbf{\gamma}' &= \mathbf{R} \mathbf{\gamma} \mathbf{R}^T 
\end{align*}
\]

Find \( \sigma_x', \sigma_y', \text{ and } \gamma_{xy}' \) w.r.t. the original \( \sigma_x, \sigma_y, \text{ and } \gamma_{xy} \).

Rotate \( x-y \) to \( x'-y' \) by an angle \( \theta \), cut the free body.

\[
Z \mathbf{F}_x' = 0
\]

\[
Z \mathbf{F}_y' = 0
\]

Rotation of Coordinate Axes (2D)

(continue)
Rotation of Coordinate Axes (2D)

- (continue)

The above calculation of $\sigma_x'$, $\sigma_y'$, and $\tau_{xy}'$ is quite tedious, and can be simplified by using the trigonometric relation

Use of Trigonometric Relation

- The above calculation of $\sigma_x'$, $\sigma_y'$, and $\tau_{xy}'$ is quite tedious, and can be simplified by using the trigonometric relation
Mohr’s Circle (2D Plane Stress)

- The above expression of $\sigma'_x$, $\sigma'_y$, and $\tau_{xy}'$ can be represented elegantly by a simple graphical representation for the plane stress. This is first invented by a German engineer Mohr.

$$e = \frac{\sigma_x + \sigma_y}{2}$$

and,

$$r = \left(\frac{\sigma_x - \sigma_y}{2} + \tau_{xy}\right)^{1/2}$$

Note: $c$ and $r$ are the center and radius of the Mohr’s circle, respectively.

More about the Mohr’s Circle

- Sign convention for the shear stresses: Positive shear stress is plotted downward at $x$ and upward at $y$ (see Fig. 4-17 (a) and (b) in the previous viewgraph). On the contrary, negative shear stress is plotted upward at $x$ and downward at $y$.

- Note that the sign convention used in the Hosford textbook is opposite to the above convention. In this class, we adopt the above usual convention, and please ignore the one used in the textbook.

- In real coordinate system, a rotation of $\theta$ (in both clockwise and counter-clockwise) means a rotation of $2\theta$ in the corresponding Mohr’s circle.
Principal Stresses and Directions

- Examining the Mohr’s circle, the “initial” state of stresses (i.e., the xy line) can be rotated to the horizontal line (either clockwise or counter-clockwise, depending on which one is smaller); at which, the normal stresses reach maximum values and shear stresses are zero (see next page). These normal stresses are known as the principal stresses, $\sigma_{p1}$ and $\sigma_{p2}$, and the planes on which they act are the principal planes.
- The rotation to either principal stresses or maximum shear from the initial state of stress is independent of the coordinate axes chosen, provided that the coordinates follow the right-hand rule.
- If the material is prone to fail by tensile cracking, it will do so by cracking along the principal planes when the value of $\sigma_{p1}$ exceeds the tensile strength of the material.

Principal Stresses and Directions

- The rotation angles required to change the state of stresses from the initial locations to the principal planes is called the principal direction.

\[
\tan 2\theta_p = \frac{\tau_{xy}}{(\sigma_x - \sigma_y)/2}
\]

\[
\sigma_{p1, p2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}
\]

**FIGURE 3.16** Principal stresses on Mohr’s circle.
Principal (or Maximum) Shear Stresses

- In the Mohr’s circle, when the initial state of stresses (i.e., the xy line) is rotated to become vertical, the shear stresses become maximum, known as the principal (or maximum) shear stresses.
- The magnitude of the principal (or maximum) shear stresses equal to the radius of the circle.

\[ \tau_{\text{max}} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_0^2} = \frac{\sigma_{\text{max}} - \sigma_{\text{min}}}{2} \]

Example

**Example 3.7**

It is instructive to use a Mohr’s circle construction to predict how a piece of blackboard chalk will break in torsion, and then verify it in practice. The torsion produces a state of pure shear, as shown in Fig. 3.15, which causes the principal planes to appear at ±45° to the chalk’s long axis. The crack will appear transverse to the}

**Figure 3.15** Mohr’s circle for simple torsion.
Example 4.1 We consider a thin sheet pulled in its own plane so that the stress components with respect to the xy axes are as given in Fig. 4.1a. We wish to find the stress components with respect to the ab axes which are inclined at 45° to the xy axes. Using the foregoing steps, we lay out the points x and y and construct Mohr’s circle, as shown in Fig. 4.1b. The ab diameter is located at $2(45^\circ) = 90^\circ$ from the xy diameter. The stress components with respect to the ab axes could be read off directly from an accurately scaled diagram. Alternatively, we can use the geometry of the diagram to calculate as follows:

\[
2\theta_t = \tan^{-1}\frac{40}{30} = 53.2^\circ
\]

\[
r = [(30)^2 + (40)^2]^{1/2} = 50 \text{ MN/m}^2
\]

\[
\sigma_x = 80 + 50 \cos (90^\circ - 53.2^\circ) = 120 \text{ MN/m}^2
\]

\[
\sigma_y = 80 - 50 \cos (90^\circ - 53.2^\circ) = 40 \text{ MN/m}^2
\]

\[
\tau_{ab} = -50 \sin (90^\circ - 53.2^\circ) = -30 \text{ MN/m}^2
\]

Example

- The figure illustrates a failure of a 15 mm diameter copper water pipe due to excess pressure from freezing. It is interesting to note that in the cross section on the right, the failure occurred on a plane inclined 45° to the tube surface, which is the plane of the maximum shear stress.
Transformation of Axes (3D)

For equilibrium requires,

\[ \Sigma E_x = 0 : \quad F_x = 0 = \frac{\sigma_{xx}}{ABC} + \gamma_{xy} \frac{\sigma_{yy}}{ABC} + \gamma_{xz} \frac{\sigma_{zz}}{ABC} \]

Introducing a segment \( \overline{CD} \), to which \( \overline{CD} \parallel ABC \), as follows,

Note the angles of \( \overline{CD} \parallel x, y, z \) axes are \( \alpha, \beta, \gamma \), respectively.

From the volume of tetrahedron

\[ \text{Volume} = \frac{1}{3} \overline{OD} (\sigma ABC) = \frac{1}{3} \overline{OA} (\sigma ABC) = \frac{1}{3} \overline{DB} (\sigma ABC) \]
Transformation of Axes (3D)

• (continue)

If shear stress on the inclined ("cut") plane were zero, this indicates the plane is "principal plane," with shear stresses, leaving only normal stresses in 3 orthogonal directions. Thus, I is normal to the plane, and can be expressed as \( I = \sigma \hat{n} \) (\( \hat{n} \): the normal vector).
Transformation of Axes (3D)

• (continue)

The above matrix becomes:

\[ \begin{pmatrix} \sigma_x & \sigma_y & \sigma_z \\ \sigma_y & \sigma_z & \sigma_x \\ \sigma_z & \sigma_x & \sigma_y \end{pmatrix} \]

\[ \begin{pmatrix} \sigma_x' \\ \sigma_y' \\ \sigma_z' \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \end{pmatrix} \]

Note that \( \sigma_p \) has 3 values, representing the principal stresses \( \sigma_1, \sigma_2, \sigma_3 \) (if \( \sigma_3 \) is assigned to 0, it becomes a 2-D problem).

The above \( \sigma_p \) equation can be simplified to:

\[ \sigma_p^2 - I_1 \sigma_p + I_2 \sigma_p - I_3 = 0 \]

where \( I_1, I_2, I_3 \) are invariants, i.e., do not change with any rotation angle, since \( \sigma_p \) are “unique” (i.e., having only one set of solutions).

Transformation of Axes (3D)

• (continue)

also notes:

\[ I_1 = \sigma_1 + \sigma_2 + \sigma_3 \]
\[ I_2 = \sigma_1 \sigma_2 + \sigma_2 \sigma_3 + \sigma_3 \sigma_1 - \frac{\sigma_1 \sigma_2}{\sigma_3} - \frac{\sigma_2 \sigma_3}{\sigma_1} - \frac{\sigma_3 \sigma_1}{\sigma_2} \]
\[ I_3 = \frac{\left| \begin{array}{ccc} \sigma_1 & \sigma_2 & \sigma_3 \\ \sigma_2 & \sigma_3 & \sigma_1 \\ \sigma_3 & \sigma_1 & \sigma_2 \end{array} \right|}{\sigma_1 \sigma_2 \sigma_3} = \det \sigma \]

\[ \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{pmatrix} \]
Transformation of Axes (3D)

• (continue)

To determine the principal axes, the method is:
For each $a_i$ (i.e., $a_1, a_2, a_3$), solve:

$$(a_i-a^*)u_i + T_{xy} + T_{xz} + T_{zx} + T_{yy} = 0$$

$$(a_i-a^*)v_j + T_{xz} + T_{xy} + T_{zy} = 0$$

$$(a_i-a^*)w_k + T_{zx} + T_{xy} + T_{yz} = 0$$

Note: $u^2 + v^2 + w^2 = 1$

Example

• (continue)

Ex.: Determine the principal stresses for the following state of stress:

\[
\begin{bmatrix}
1 & 1 & 1 \\
1 & 2 & 1 \\
1 & 1 & 3
\end{bmatrix}
\]

Set:

$$\sigma^3 = I_3 \sigma^3 + I_2 \sigma^2 + I_1 \sigma - I_0 = 0$$

$$I_3 = \sigma_1 + \sigma_2 + \sigma_3 = 3$$

$$I_2 = \frac{1}{2} \left( \sigma_1 \sigma_2 + \sigma_2 \sigma_3 + \sigma_3 \sigma_1 \right) - \frac{1}{3} \sigma_3^2 = 0$$

$$I_1 = \frac{1}{6} \left( \sigma_1^2 + \sigma_2^2 + \sigma_3^2 \right) - \frac{1}{3} \sigma_3^2 = 0$$

$$\sigma_3^3 - 3 \sigma_3^2 \sigma_0 = 0$$

Thus, $\sigma_0 = 0, 0.3, \text{ and } \infty$.

Max: $\sigma_0 = 15, 15, 0 \text{ MPa}$

Note: $\sigma_0 = 0$ to the origin.
• (continue)

Example

For principal axes, take $c = 3$.

\[
\begin{bmatrix}
-3 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & 1
\end{bmatrix}
= \begin{bmatrix}
-2 & 0 & 0 \\
0 & -2 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

\[
\begin{bmatrix}
-2 & 0 & 0 \\
0 & -2 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\cos\alpha & \cos\beta & \cos\gamma \\
\sin\alpha & \sin\beta & \sin\gamma
\end{bmatrix}
= 0
\]

\[
\begin{bmatrix}
-2 \cos\alpha + \cos\beta + \cos\gamma \\
\cos\alpha - 2 \cos\beta + \cos\gamma \\
\cos\alpha + \cos\beta - 2 \cos\gamma
\end{bmatrix}
= 0
\]

also consider $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.

• (continue)

Example

2. When $c \neq 0$:

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\cos\alpha & \cos\beta & \cos\gamma \\
\sin\alpha & \sin\beta & \sin\gamma
\end{bmatrix}
= 0
\]

\[
\begin{bmatrix}
\cos\alpha + \cos\beta + \cos\gamma \\
\cos\alpha - 2 \cos\beta + \cos\gamma \\
\cos\alpha + \cos\beta - 2 \cos\gamma
\end{bmatrix}
= 0
\]

also consider $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.

3. $\mathbf{e} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$\mathbf{y} = \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$

$\mathbf{z} = \mathbf{e} \times \mathbf{y} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$\mathbf{c} = \mathbf{e} \times \mathbf{z} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$\mathbf{c} \times \mathbf{z} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$
Example

Well, how do we verify the above result? By rotation of axes.

\[ T_x x = 0x_x (x' x') + 0y_y (x' y') + T_z z (x' z') \]
\[ T_y y = 0x_x (y' x') + 0y_y (y' y') + T_z z (y' z') \]
\[ T_z z = 0x_x (z' x') + 0y_y (z' y') + T_z z (z' z') \]

Note: the \((x', y', z')\) are transformed coordinates, e.g., \(x' = \cos \theta\).

Example

(continue)

in matrix form:

\[
\begin{bmatrix}
T_x & T_y & T_z \\
T_y & T_y & T_z \\
T_z & T_z & T_z
\end{bmatrix}
= \begin{bmatrix}
x'(x') & y'(x') & z'(x') \\
x'(y') & y'(y') & z'(y') \\
x'(z') & y'(z') & z'(z')
\end{bmatrix}
\]

Similarly:

\[
\begin{bmatrix}
T_y & T_y & T_z \\
T_y & T_y & T_z \\
T_z & T_z & T_z
\end{bmatrix}
= \begin{bmatrix}
x'(y') & y'(y') & z'(y') \\
x'(y') & y'(y') & z'(y') \\
x'(z') & y'(z') & z'(z')
\end{bmatrix}
\]

Hence:

\[
\begin{bmatrix}
T_x & T_y & T_z \\
T_y & T_y & T_z \\
T_z & T_z & T_z
\end{bmatrix}
= \begin{bmatrix}
(x'(x')) & (y'(x')) & (z'(x')) \\
(x'(y')) & (y'(y')) & (z'(y')) \\
(x'(z')) & (y'(z')) & (z'(z'))
\end{bmatrix}
\]

\[
T_z T_y T_z = \begin{bmatrix}
(x'(x')) & (y'(x')) & (z'(x')) \\
(x'(y')) & (y'(y')) & (z'(y')) \\
(x'(z')) & (y'(z')) & (z'(z'))
\end{bmatrix}
\]

\[
T_x T_y T_z = \begin{bmatrix}
(x'(x')) & (y'(x')) & (z'(x')) \\
(x'(y')) & (y'(y')) & (z'(y')) \\
(x'(z')) & (y'(z')) & (z'(z'))
\end{bmatrix}
\]

\[
T_y T_y T_z = \begin{bmatrix}
(x'(x')) & (y'(x')) & (z'(x')) \\
(x'(y')) & (y'(y')) & (z'(y')) \\
(x'(z')) & (y'(z')) & (z'(z'))
\end{bmatrix}
\]
Example

• (continue)

Similarly, \( T'x', T'y', T'z' \) can be expressed as \( Tx, Ty, Tz \)

\[
\begin{bmatrix}
T'x' \\
T'y' \\
T'z'
\end{bmatrix} =
\begin{bmatrix}
T_x & T_y & T_z \\
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix}
\]

Also for \( [T'x', T'y', T'z'] \) and

\[
[T_x, T_y, T_z]
\]

(not shown here for simplicity)

Overall in 3-D

\[
\begin{bmatrix}
T'x' \\
T'y' \\
T'z'
\end{bmatrix} =
\begin{bmatrix}
(x'c) & (x's) & (x'b) \\
(y'c) & (y's) & (y'b) \\
(z'c) & (z's) & (z'b)
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix}
\]

This is the rule of axes rotation.

Example

• (continue)

We now verify the previously obtained direction cosines by the rotation of axes.

\[
\begin{bmatrix}
\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\
\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\
\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}}
\end{bmatrix}
\begin{bmatrix}
\frac{1}{\sqrt{3}} \\
\frac{1}{\sqrt{3}} \\
\frac{1}{\sqrt{3}}
\end{bmatrix}
\]

\[
= \begin{bmatrix}
\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\
\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\
\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}}
\end{bmatrix}
\]

\[
\begin{bmatrix}
\frac{1}{\sqrt{3}} \\
\frac{1}{\sqrt{3}} \\
\frac{1}{\sqrt{3}}
\end{bmatrix}
\]

\[
\]

So, the direction cosines are valid solutions.
Example

Example problem 1.2: Find the principal stresses in a body under the stress state \( \sigma_x = 10, \sigma_y = 8, \sigma_z = -5, \tau_{yx} = \tau_{zy} = 5, \tau_{zx} = \tau_{xz} = -4, \) and \( \tau_{xy} = \tau_{yx} = -8, \) where all stresses are in MPa.

SOLUTION: Using Equation (1.13), \( I_1 = 10 + 8 - 5 = 13, I_2 = 5^2 + (-4)^2 + (-8)^2 - 8(-5) - (-5)10 - 10(-8) = 115, \) and \( I_3 = 10 \cdot 8(-5) + 2 \cdot 5(-4)(-8) - 10 \cdot 5^2 - 8(-4)^2 - (-5)(-8)^2 = -138. \)

Solving Equation (1.11) gives \( \sigma_p^3 - 13\sigma_p^2 - 115\sigma_p + 138 = 0, \) \( \sigma_p = 1.079, 18.72, -6.82. \)

Methods to Solve Cubic Equations

• Solutions to solve cubic equations are available by referring to Handout 1 of the class. The references are from the following mathematical handbooks:

Application of Transformation of Axes (3D)

• In our textbook, the state of stresses in the “new” coordinate system has also related to the “old” (or original) coordinate system in tensor form. This operation is very useful when we want to find shear stresses on a certain slip system when a material is subjected to an external stress acting on a crystal.

• In tensor form: \( \sigma_{ij} = l_{im} l_{jn} \sigma_{mn} \)
  where \( l_{im} \) is the cosine of the angle between the \( i \) and \( m \) axes.

Example

Example problem 1.1: A cubic crystal is loaded with a tensile stress of 2.8 MPa applied along the [210] direction as shown in Figure 1.6. Find the shear stress on the (111) plane in the [101] direction.

SOLUTION: In a cubic crystal, the normal to a plane has the same indices as the plane, so the normal to (111) is [111]. Also, in a cubic crystal, the cosine of the angle between two directions is given by the dot product of unit vectors in those directions. For example, the cosine of the angle between \([u_1, v_1, w_1]\) and \([u_2, v_2, w_2]\) is equal to \((u_1 w_2 + v_1 v_2 + w_1 w_2) / \sqrt{(u_1^2 + v_1^2 + w_1^2)(u_2^2 + v_2^2 + w_2^2)})^{1/2}.

Designating [210] as \( x \), [101] as \( d \), and [111] as \( n \), \( \tau_{sd} = l_{sx} l_{dx} \sigma_{xx} = ((2 - 1 + 1 + 0 - 1) / \sqrt{(2^2 + 1^2 + 0) (1^2 + 1^2 + 1^2)}) \cdot ((2 - 1 + 1 + 0 - 1) / \sqrt{(2^2 + 1^2 + 0) (1^2 + 1^2 + 1^2)}) \cdot 2.8 \text{ MPa} = 2.866 / 3.16 MPa = 0.908 \text{ MPa}.\)
Mohr’s Circles for 3-D (Example)

Strain

• An engineering normal strain is defined as the change of length divided by the original length, i.e.
  \[ e = \frac{\Delta L}{L_0} \]
  
• Unit of strain: None (or unit-less)

• Definition of shear strain (γ):
  We can define shear strain exactly the way we do longitudinal strain: the ratio of deformation to original dimensions. In the case of shear strain, though, it’s the amount of deformation perpendicular to a given line rather than parallel to it. The ratio turns out to be tan γ, where γ is the angle the sheared line makes with its original orientation.
Strain

- Strain is also a symmetric second-order tensor, identical to the stress. Therefore, there are 6 independent variables in the strain matrix, instead of 9.
- Strain can also be "rotated" to find its principal strain, principal strain direction, and maximum shear strain. The operation, including the Mohr's strain circle, is very similar to that of stress.
- But, the only difference is that changes need to be made when the subscripts are different, i.e., when $i \neq j$.

There exists a $\frac{1}{2}$ difference when $i \neq j$. The $\frac{1}{2}$ is because $\gamma_{xy} = \varepsilon_{xy} + \varepsilon_{yx} = 2 \varepsilon_{xy}$.

Axis Transformation for Strains

- Principal strains, principal directions, and maximum strains can be found from any given state of strain following the same procedure as that for the stress, so long as the $\frac{1}{2}$ difference when $i \neq j$ has being taken care of.

\[
\begin{bmatrix}
  (\varepsilon_x - \varepsilon) & \frac{\gamma_{xy}}{2} & \frac{\gamma_{xz}}{2} \\
  \frac{\gamma_{xy}}{2} & (\varepsilon_y - \varepsilon) & \frac{\gamma_{yz}}{2} \\
  \frac{\gamma_{xz}}{2} & \frac{\gamma_{yz}}{2} & (\varepsilon_z - \varepsilon)
\end{bmatrix} = 0
\]

Solve the cubic equations for the strains.
Boundary Conditions

- Simplifying the analysis of mechanical problems.
- Some most important ones are listed as follows:
  1. Free surfaces: On a free surface, the two shear stresses (i.e., $\tau_{xy} = \tau_{yx} = 0$ if $z$ is the normal to the free surface) in the surface vanish. Unless there is a pressure acting on the free surface, the stress normal (i.e., $\sigma_{zz} = 0$) also vanishes.
  2. Constraints from neighboring regions: Deformation is compatible, e.g., $\varepsilon_{xA} = \varepsilon_{xB}$ in the figure below.
  3. St. Venant’s principle: The restraint from any end or edge effect will disappear after one characteristic length.

Summary

- Definition and calculation of simple stress and strain, including the normal and shear ones.
- Calculation of principal stresses/strains, principal directions, and maximum shear stresses/strain. This operation allows one to evaluate which material is best suited for the application.
- For plane stress condition, use of Mohr’s circle to estimate the above mechanical properties.