Introduction

• As the loading stress exceeds yield strength of the material, plastic deformation occurs for *ductile* materials. Plasticity theory deals with yielding of materials under complex stress state. Development of *yielding criteria* is hence pivotal in predicting whether or not a material will begin to yield under a given stress state, resulting in shape change or even failure of the specimen. Hence, the yield criteria are critically important in materials design and selection in practice.

• For *brittle* materials, a *fracture criterion* based upon the maximum normal stress is generally used.
Yield Criteria

- A yield criterion is a mathematical expression of the stress states that will cause yielding or plastic flow. Note that crystalline materials yield or deform under shear stresses by slip and twinning mechanisms.
- There are two criteria to describe the yielding of crystalline solids.
  - Tresca criterion: simple, yielding will occur when the largest shear stress reaches a critical value.
  - von Mises criterion: effect of principal stresses is included in the determination of the onset of yielding occurrence.

Assumes materials are isotropic.

Tresca Criterion

- Tresca criterion states that yielding will occur when the largest shear stress reaches a critical value. The largest shear stress is \( \tau_{\text{max}} = \frac{\sigma_{\text{max}} - \sigma_{\text{min}}}{2} \), so the Tresca criterion can be expressed as

\[
\sigma_{\text{max}} - \sigma_{\text{min}} = C \quad (C: \text{A critical value})
\]

In terms of the Mohr’s stress circle in three dimensions, when the principal stresses are (for example) \( \sigma_1 \geq \sigma_2 \geq \sigma_3 \), the Tresca criterion is then expressed as

\[
\sigma_1 - \sigma_3 = C
\]

The constant C can be found by considering uniaxial tension.

Note: The use of numerical subscript represents principal stresses.
Tresca Criterion

- In a uniaxial tension test, $\sigma_2 = \sigma_3 = 0$ and at yielding $\sigma_1 = \sigma_Y$. Now substituting this to the previous equation, we find $C = \sigma_Y$. Therefore, the Tresca criterion may be expressed as

$$\sigma_1 - \sigma_3 = \sigma_Y = Y$$

When $\sigma_{\text{max}} - \sigma_{\text{min}} \geq Y$ yielding occurs.

When, $\sigma_{\text{max}} - \sigma_{\text{min}} < Y$ yield will not occur.

Example: For pure shear, $\sigma_1 = -\sigma_3$, when does the material yield under Tresca.

Sol.: The Tresca criterion becomes

$$\sigma_1 - \sigma_3 = 2\sigma_1 = Y$$

:. $\text{when } \sigma_1 \geq \frac{Y}{2}$ yielding occurs

Example problem 6.4: Consider an isotropic material, loaded so that the principal stresses coincide with the $x$, $y$, and $z$ axes of the material. Assume that the Tresca yield criterion applies. Make a plot of the combinations of $\sigma_x$ versus $\sigma_y$ that will cause yielding with $\sigma_z = 0$.

**SOLUTION:** Divide the $\sigma_x$, $\sigma_y$, stress space into six sectors as shown in Figure 6.2. The following conditions are appropriate:

1. $\sigma_y > \sigma_z > \sigma_x = 0$, so $\sigma_1 = \sigma_x$, $\sigma_3 = \sigma_z = 0$, so $\sigma_2 = \sigma_Y$
2. $\sigma_y > \sigma_z > \sigma_x = 0$, so $\sigma_1 = \sigma_z$, $\sigma_3 = \sigma_y = 0$, so $\sigma_2 = \sigma_Y$
3. $\sigma_y > \sigma_z > 0 = \sigma_x$, so $\sigma_1 = \sigma_y$, $\sigma_3 = \sigma_z > 0$, so $\sigma_2 = \sigma_Y$
4. $\sigma_y = 0 > \sigma_z > \sigma_x = 0$, so $\sigma_1 = \sigma_z = 0$, $\sigma_3 = \sigma_y = \sigma_Y$
5. $\sigma_z = 0 > \sigma_x > \sigma_y = 0$, so $\sigma_1 = \sigma_y$, $\sigma_3 = \sigma_z = 0$, so $\sigma_2 = \sigma_Y$
6. $\sigma_x > \sigma_z > 0 = \sigma_y$, so $\sigma_1 = \sigma_x$, $\sigma_3 = \sigma_z = 0$, so $\sigma_2 = \sigma_Y$

These are plotted in Figure 6.2.

Inside the hexagon, no yield.

Outside the hexagon, yield occurs.
Example

From the Mohr's circle

\[ \tau = \frac{Y}{2} \text{ (or } k/2 \text{ in the graph)} \]

\[ \therefore Y^2 = \left(\frac{\sigma}{2}\right)^2 + \left(\frac{Y}{2}\right)^2 \]

\[ \sigma = \frac{\sqrt{3}}{2} Y \]

Tresca Criterion in 3D

- For isotropic materials under 3D state of stresses, a tube with a hexagonal cross section as shown below. The axis of the tube is the line \( \sigma_1 = \sigma_2 = \sigma_3 \).
von Mises Criterion

- The effect of the intermediate principal stress can be included by assuming that yielding depends on the root-mean-square diameter of the three Mohr’s circle. This is the von Mises criterion, which can be expressed as

\[
\sqrt[3]{\frac{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}{3}} = C
\]

Note that each term is squared, so the convention \( \sigma_1 \geq \sigma_2 \geq \sigma_3 \) is not necessary. \( C \) can be evaluated by considering a uniaxial tension test. At yielding, \( \sigma_1 = Y \) and \( \sigma_2 = \sigma_3 = 0 \). Substituting into the above equation,

\[
\frac{Y^2 + 0^2 + (-Y)^2}{3} = C^2 \quad \text{or} \quad C = \left(\frac{2}{3}\right)^{1/2} Y
\]

Therefore, yielding occurs when

\[
(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \geq 2Y^2
\]

von Mises Criterion

- For a state of pure shear in isotropic mater., \( \sigma_1 = -\sigma_3 \) and \( \sigma_2 = 0 \). Substituting into the von Mises equation,

\[
(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = \sigma_1^2 + \sigma_2^2 + \sigma_3^2 + [(-\sigma_1) - \sigma_1]^2 = 6\sigma_1^2 = 2Y^2
\]

\[\therefore \text{ when } \sigma_1 = -\sigma_3 \geq \frac{Y}{\sqrt{3}} \text{ yielding occurs}\]

- In case of plane stress, one of the principal stresses is zero. Therefore, substituting \( \sigma_3 = 0 \), the equation becomes

\[
(\sigma_1 - \sigma_2)^2 + \sigma_2^2 + \sigma_1^2 = \frac{2}{2} \sigma_1^2 - 2\sigma_1 \sigma_2 + 2\sigma_2^2 = 2Y^2
\]

This equation reveals that the dependence between yield stress and principal stresses is an ellipse.
Examples

- Plane stress situation.

Let

\[ \alpha = \frac{\sigma_2}{\sigma_1} \]

\[ \therefore \sigma_1^2 - \sigma_2 + \sigma_2^2 \]

\[ = \sigma_1^2 - \alpha \sigma_1^2 + \alpha^2 \sigma_2^2 \]

\[ = \sigma_1^2(1 - \alpha + \alpha^2) = Y^2 \]

\[ \sigma_1 = \frac{Y}{\sqrt{1 - \alpha + \alpha^2}} \]

Sol.

\[ (1 - \alpha + \alpha^2)^{1/2} \text{ value is the smallest for largest } \sigma_1/Y \text{ ratio.} \]

\[ \frac{d(1 - \alpha + \alpha^2)}{d\alpha} = 1 + 2\alpha = 0 \]

\[ \alpha = -0.5 \]

\[ \left( \frac{\sigma_1}{Y} \right)_{\text{max}} = \frac{1}{\sqrt{-0.5 + 0.5Y}} = 1.155 \]

Comparison of Yield Criteria for Plane Stress (Isotropic Materials)

- The yield locus for the Tresca criterion falls inside of the von Mises yield ellipse. Note that the two yielding criteria predict the same yield stress for conditions of uniaxial stress and balanced biaxial stress (\( \sigma_1 = \sigma_2 \) or \( \sigma_3 \)). The greatest divergence (15.5% at \( \alpha = 0.5 \)) between the two occurs for pure shear.
Example of Isotropic Yielding

• A number of ideas have been set forth attempting to rationalize the von Mises condition on fundamental grounds. However, this is essentially an empirical criterion that more accurately describes yielding under multiaxial stress states than does the Tresca condition. As shown in the underneath figure, experimentally obtained from biaxial yielding are shown.

Examples

• Yield locus diagram for Ti-Al(4%)-O₂(0.25%) at 0.2, 1, and 4% strains for both tensile and microharness indentation (Knoop) testings, revealing good fit when compared to von Mises criterion.
Remarks

• The von Mises yield criterion can also be expressed in terms of stresses that are not principal stresses. In this case, shear terms are included such that

$$(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2) = 2Y^2$$

where $x$, $y$, and $z$ are not principal stress axes.

von Mises Criterion in 3D

• For the general case, where all three principal normal stresses may have non-zero values, the boundary of the region of no yielding represents a circular cylindrical surface with its axis along the line $\sigma_1 = \sigma_2 = \sigma_3$. If any one of $\sigma_1$, $\sigma_2$, $\sigma_3$ or is zero, the intersection of the cylindrical surface with the plane of the remaining two principal stresses gives an ellipse.
Other Isotropic Yield Criteria

• von Mises and Tresca are two most frequently used yield criteria, but are not the only possible isotropic criteria. Other criteria tend to lie between the two and can be represented by

\[ |\sigma_1 - \sigma_2|^a + |\sigma_2 - \sigma_3|^a + |\sigma_3 - \sigma_1|^a = 2Y^a \]

This criterion reduces to von Mises for \( a = 2 \) and \( a = 4 \), and to Tresca for \( a = 1 \) and \( a \to \infty \). For exponent values greater than 4, this criterion predicts yield loci between Tresca and von Mises (see next viewgraph).

Yield Loci

• Yield loci for \( a \) of different values. Note that the von Mises criterion corresponds to \( a = 2 \) and the Tresca criterion to \( a = 1 \).

Theoretical calculation based on \{111\}<110> slip system suggests an exponent of \( a = 8 \) for fcc metals. Similar calculation suggests that \( a = 6 \). These values fit experimental data as well.
Anisotropic Plasticity

- Although it is frequently assumed that materials are isotropic, they rarely are. There are two main causes of anisotropy. One cause is preferred orientation of grains or crystallographic texture. The other is mechanical fibering, which is the elongation and alignment of microstructural feature such as inclusions and grain boundaries.
- If the loading is such that the directions of principal stress coincide with the symmetry axes and if there is planar isotropy, a generalization of the von Mises criterion is

\[ R(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 = (R + 1)X^2 \]

where \( X \) is the yield strength in uniaxial tension.

Example of Anisotropic Yielding

- Yield locus for highly textured titanium alloy sheet. Note that the experimentally determined curve is nonsymmetric when compared with the ideal isotropic curve.
Anisotropic Plasticity

- Plane stress ($\sigma_z = 0$) yield locus predicted by Hill (1948) for anisotropic plasticity with several values of $R$. The dashed line is the locus of stress states that produce plane strain ($\varepsilon_y = 0$).

Anisotropic Plasticity

- The Hill model often over-estimate the effect of $R$-value on the flow stress. Hosford modified Hill’s model by changing the exponent from 2 to $a$. Comparison Hosford’s model with that of Hill.

The yield loci of $a = 2$ correspond to Hill’s model.
Deformation of Polymers

- The von Mises criterion can also be extended to deformation of polymers, even though the plastic deformation of polymers often involve formation of craze, instead of slip bands and dislocations in metals and some ceramics.

- Tension promotes yielding and compression delay it.

Strain-hardening on Yield Locus

- **Isotropic hardening model:** The effect of strain-hardening is simply to expand the yield locus without changing its shape. The stresses for yielding are increased by the same factor along all loading paths, and can be applied to anisotropic materials also.

- **Kinematic hardening model:** Plastic deformation simply shifts the yield locus in the direction of the loading path without changing its shape or size. If the shift is large enough, unloading may actually cause plastic deformation.
Summary

- Design to avoid yielding or fracture in materials without existing microcracks requires the use of a failure criterion. Two yield criteria are available that are reasonably accurate for isotropic materials, i.e., the Tresca and the von Mises criteria.

\[
\text{Tresca:} \quad \text{Max}(|\sigma_1 - \sigma_2|, |\sigma_2 - \sigma_3|, |\sigma_3 - \sigma_1|) = \sigma_Y = Y
\]

\[
\text{von Mises:} \quad (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = 2Y^2 (= 2\sigma_Y^2)
\]

- The yield criteria can reasonably apply to ductile materials, but there is no single failure criterion suffices to describe the fracture behavior for brittle materials.

Failure Locus of a Brittle Mater.

- For an isotropic materials under plane-stress situation \((s_3 = 0)\), a graphical representation of the fracture criterion is shown in square. Note that no single failure criterion suffices to describe the fracture behavior for all brittle materials.

\[
\sigma_u \text{ is the stress at which the material fails.}
\]

Within the square, the material is safe to use.

The ratio of \(D_N/D\) is defined as the “safety factor.”

\[
\text{The square becomes a CUBE when } \sigma_3 \text{ not equals to zero.}
\]

Return